1 INTRODUCTION

Graph theory is the study that studies the structure of a network, and it is used for finding communities in networks where it is desired to detect hierarchies of substructures, it is also used for ranking (ordering) hyperlinks or by GPS to find the shortest path. This chapter is made essentially to give a brief background about graph theory and then dive in the explanation of the minimum weight dominating set problem.

2 Definition and Fundamental Concepts

2.1 Graph Theory

2.1.1 Definition

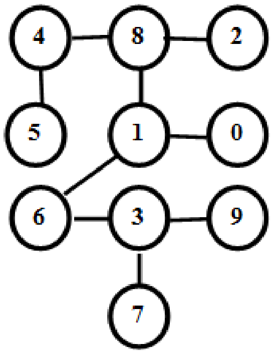
The graphs are usually represented by diagrams, in which the vertices are points. An edge x, y is shown as a line from (the point representing) x to (the point representing) y. To distinguish the vertices from other points in the plane, they are often drawn as small circles or large dots [15]

Generally, vertices and edges connecting the vertices forms a graph. Formally, a graph is a pair of sets (V, E), where V is the set of vertices and E is the set of edges, formed by pairs of vertices.

Vertices: nodes or points in the graph.

Edge: an edge is a path (bridge, line) between two regions or a relation between two objects.

The graph can be represented with three ways as a drawing representation or as adjacency matrix representation or as an adjacency list representation as shown in the next page:



v	0	1	2	3	4	5	6	7	8	9
0	0	1	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	1	0	1	0
2	0	0	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	1	0	0
4	0	0	0	0	0	1	0	0	1	0
5	0	0	0	0	1	0	0	0	0	0
6	0	1	0	1	0	0	0	0	0	0
7	0	0	0	1	0	0	0	0	0	0
8	0	1	1	0	1	0	0	0	0	0
9	0	0	0	1	0	0	0	0	0	0

v	Adj_v
0	{1}
1	{0,6,8}
2	{8}
3	{6,7,9}
4	{5,8}
5	{4}
6	{1,3}
7	{3}
8	{1,2,4}
9	{3}

1-An undirected graph 2-Adjacency matrix representation 3-Adjacency list representation

Figure 2.1- Graph representation

2.1.2 Terminologies

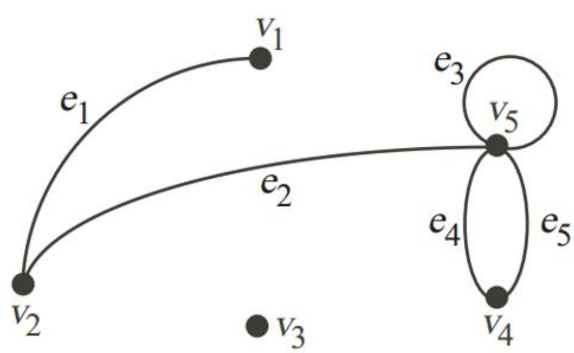


Figure 2.2- Graph with label edges [2]

We have the following terminologies [16]:

1. The two vertices u and v are end vertices of the edge (u, v): (v4 and v5 are end vertices of e5.)
2. Edges that have the same end vertices are parallel: (e4 and e5 are parallel.)
3. An edge of the form (v, v) is a loop: (e3 is a loop.)
4. A graph is simple if it has no parallel edges or loops: (The graph is not simple.)
5. A graph with no edges (i.e. E is empty) is empty.

6. A graph with no vertices (i.e. V and E are empty) is a null graph.
7. A graph with only one vertex is trivial.
8. Edges are adjacent (neighbors) if they share a common end vertex: (e1 and e2 are adjacent.)
9. Two vertices u and v are adjacent if they are connected by an edge, in other words, (u, v) is an edge: (v1 and v2 are adjacent.)
10. The degree of the vertex v, written as d(v), is the number of edges with v as an end vertex. By convention, we count a loop twice and parallel edges contribute separately.
11. A pendant vertex is a vertex whose degree is 1: (The degree of v1 is 1 so it is a pendant vertex.)
12. An edge that has a pendant vertex as an end vertex is a pendant edge: (e1 is a pendant edge.)
13. An isolated vertex is a vertex whose degree is 0: (the degree of v3 is 0 so it is an isolated vertex.)

Degrees of vertices: Let G = (V, E, φ) be a graph and v ∈ V a vertex. The degree of v, d(v) is the number of e ∈ E such that v ∈ φ(e); i.e., e is incident on v. according to figure 1.2 the degree of v2 is 2 because v2 has two neighbors

2.1.3 Graph types

There are several types in the graph representation we will mention some of them in this section

Simple graph

A graph is simple if it has no loops and no two of its links join the same pair of vertices.

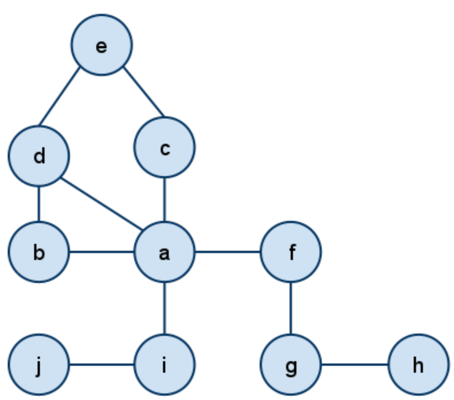


Figure 2.3- simple graph representation [3]

Directed graph

Vertices connected by directed edges, or arcs forms a directed graph or digraph. Formally, a digraph is a pair (V, E), where V is the vertex set and E is the set of vertex pairs where elements of E are ordered pairs: the arc from vertex u to vertex v is written as (u, v) .

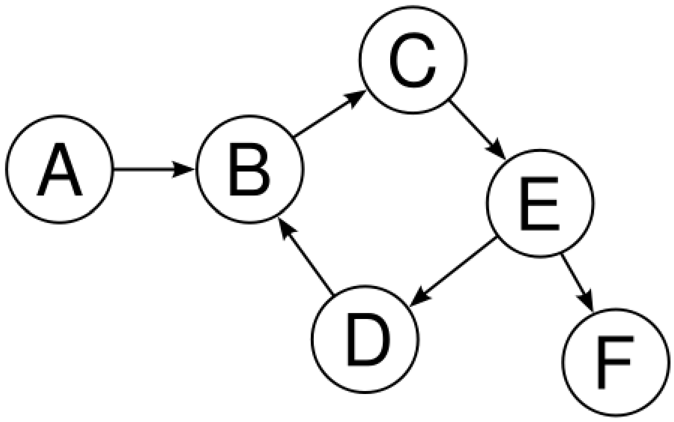


Figure 2.4- Directed Graph [4]

Undirected graph

An undirected graph is a set of objects (called vertices or nodes) that are connected together, where all the edges are bidirectional. When drawing an undirected graph, the edges are typically drawn as lines between pairs of nodes, as shown in the following figure.

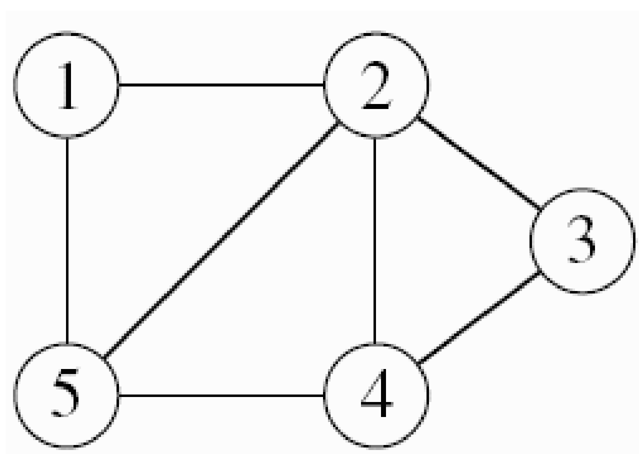


Figure 2.5- Undirected Graph [5]

Sub-graph

A sub-graph of a graph G is a graph H such that: V(H) Í V(G) and E(H) Í E(G) and the assignment of endpoints to edges in H is the same as in ***G***.

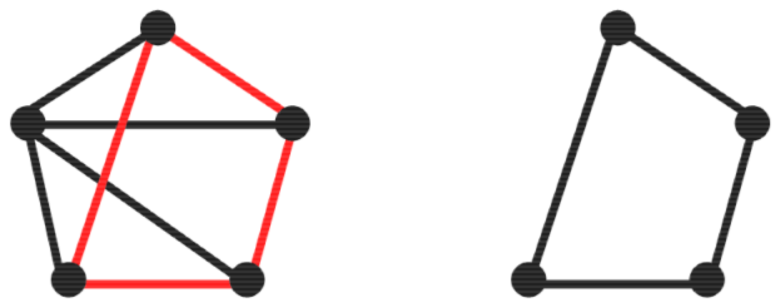


Figure 2.6- sub-graph example [6]

Complete graph

A simple graph whose vertices are pairwise adjacent, a complete graph with n vertices is denoted as K_n. The first seven complete graphs are given in the examples below:

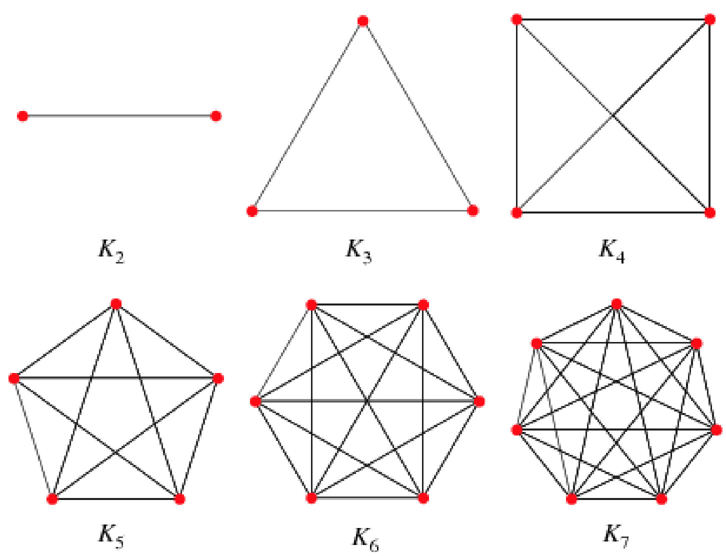
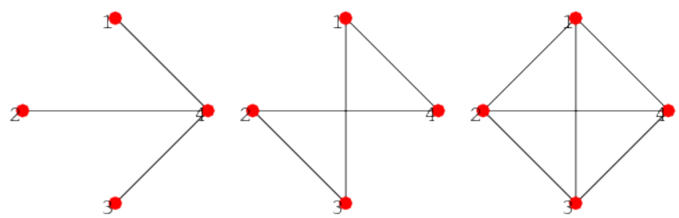


Figure 2.7- complete graph example [7]

Adjacency matrix

The adjacency matrix, of a simple labeled graph is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position (v_i,v_j) according to whether v_i and v_j are adjacent or not. For a simple graph with no self-loops, the adjacency matrix must have 0s on the diagonal.



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Figure 2.8- adjacency matrix example [8]

2.2 APPLICATION OF GRAPH

Because of its inherent simplicity, graph theory has a very wide range of applications, in engineering, in physical, social, and biological sciences, in linguistics and in numerous other areas. A graph can be used to represent almost any physical situation involving discrete object and a relationship. [17]

2.2.1 Social Network Analysis

Graph-theoretic and statistical techniques can be used to analyze some important parameters of global social networks and clarify their use in social science studies for example network analysis that is the process of capturing network traffic and scan it closely to decide what is happening on the network.

According to Maksim Tsvetovat and Alexander Kouznetsov social network analysis is:

"In a few words, Social Network Analysis (SNA) can be described as a “study of human relationships by means of graph theory.” However, this sentence leaves a lot to be unpacked...In a way, SNA is similar to many statistical methods. The fact that economists use regression analysis extensively does not mean that the technique is limited to the study of economics. Similarly, while studying the social media is a great way to apply SNA techniques—not only is the data easily available, but the opportunities for studying are numerous and lucrative." [18] Example: Most known examples are Facebook and Twitter

2.2.2 Route Planning

Computing the most cost-effective route involving several nodes or stopovers by minimizing the distance traveled and/or time taken. [19]

Logistics firms that operate their own fleets, tend to use a route plan that has the vehicles starting and ending at the same location. This ensures the minimum repositioning of vehicles and personnel. However, to develop routes that cover all deliveries and pickups to and from numerous customers is extremely complex and to develop the routes that are most efficient is becoming increasingly difficult. Many route planners can develop efficient routes but find that due to rules on how many hours a driver can operate a vehicle, such as the regulations from the US Department of Transport, mandate that a less efficient route is used. [20]

Example: Google Maps.

3 Minimum Weight Dominating Set Problem

In this section we will explqin the MWDS problem (minimum weight dominating set) problem.

3.3 Minimum Dominating set

In the example figure 2.10 –next page-, if we take the vertex A ∈ S we get all the other vertices as a neighbors to A. This parameter is called the dominating set and any dominating set with a minimum cardinality is called minimum dominating set.

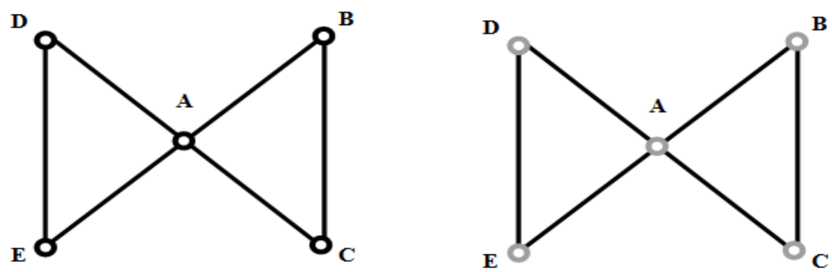


Figure 2.10- dominating set example

4 Minimum Weight Dominating Set problem (MWDS)

4.1 Definition

For given undirected graph G (V,E), where V is the set of the vertices and E is the set of edges, with weights associated to its vertices, a dominating set is a subset D ⊆ V such that every vertex v ⊆ V is either a member of D or is adjacent to a vertex in D.

A dominating set D is called the minimum dominating set if its cardinality is minimum of all possible dominating sets. This cardinality in our case is the weight so we desire to obtain a subset with minimum weight .In other word the minimum dominating set is a dominating set D, such that ∑w(v) ⊆ D is minimized. According to a function $w: V \rightarrow \mathbb{R}^{+}$ that associates a positive weight value $w(v)$ to each vertex v ⊆ V.

Before starting with the detailed explanation of the MWDS problem, we introduce the following notations and definitions.

Notation	Definition
d(v)	For a vertex $v \subseteq$ V. The degree of v is the number of v's neighbors
N[v] = N(v) ∪ {v}	the set of the neighbors of the v in G
W(D)	the set of the white vertices
d(v) := | {(v, u) ∈ E | u ∈ W(D)}|	the current D-degree of a vertex $v \in$ V \D
let W(u) := $\sum_{s\in(N(u)\cap W(S)} w(D)$	the total weight of the white neighbors of a vertex u.

Table 2.1 – notations and definitions

Each vertex in D is called a dominator; otherwise, it is called dominatee. A dominator dominates or covers itself and all its neighbors.

5 Greedy heuristic for MWDS

A greedy algorithm is about making decision incrementally in small steps without backtracking and these decisions often based on some fixed and simple priority rules. The greedy paradigm is one of the fundamental techniques in the design of algorithms. This is because it is simple, efficient, and generally do not need time consumption. (Find more in chapter 3)

For this sake of finding a better solution for the MWDS problem, than the solution found in the article [22], an Experimental comparison of two improved greedy algorithms has been performed, talking about the MWDS let's consider a undirected graph G (V,E) has a partial solution D (partial dominating set),at a particular construction step. Given D, a solution component v is selected to be included to the subset D according to one of these following heuristic functions:

1) $v \leftarrow argmax \left\{\frac{d(u)}{w(u)} + 1 \middle| u \in N(S)\right\}$. This heuristic selects a vertex u with the maximum ratio between its current degree and its weight + 1 if the u vertex is white
2) $v \leftarrow argmax \left\{\frac{W(u)+w(u)}{w(u)} \middle| u \in N(S)\right\}$.

To clarify things more let's define our input output and the function used in here

Input: an undirected, vertex-weighted graph $G = (V, E)$ with vertex weights $w \geq 0$.

Output : set of vertices that respect this condition

6 Applications of the MWDS

The MWDS has a wide range of application any many domains and according to what Sachchida Nad Chaurasia and Alok Singh mentioned in their article:

MWDS finds practical applications in diverse domains such as clustering in wireless networks, intrusion detection in adhoc networks, multidocument summarization in information retrieval, query selection in web databases. [23]

Clustering is an important mechanism for obtaining scalability and reducing energy consumption, and achieving better network results. Our algorithm can be used o cluster the nodes according to specific characteristics, for the goal of obtaining better results. Even multidocument summarization, which is a produce that takes from different texts that are talking about the same topic, and then generate a summary about all the texts, our algorithm, can be used to obtain subset with the most important word or the most appearing words, and this may facilitate the summary generation